

WEAK BANACH-SAKS PROPERTY AND KOMLÓS' THEOREM FOR PREDUALS OF JBW*-TRIPLES

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ABSTRACT. We show that the predual of a JBW*-triple has the weak Banach-Saks property, that is, reflexive subspaces of a JBW*-triple predual are super-reflexive. We also prove that JBW*-triple preduals satisfy the Komlós property (which can be considered an abstract version of the weak law of large numbers). The results rely on two previous papers from which we infer the fact that, like in the classical case of L^1 , a subspace of a JBW*-triple predual contains ℓ_1 as soon as it contains uniform copies of ℓ_1^n .

1. INTRODUCTION

From the two recent papers [22, 23], which precede this note, we know that if a sequence in the predual of a JBW*-triple spans the spaces ℓ_1^n uniformly almost isometrically then it admits a subsequence spanning ℓ_1 almost isometrically, as it is well known for L^1 -spaces. This note gathers some classical consequences of this result, namely, the weak Banach-Saks property (for which we use a criterion of Rosenthal-Beauzamy), the fact that reflexive subspaces of JBW*-triple preduals are super-reflexive, and the Komlós property. The latter generalizes the (weak) law of large numbers in L^1 (cf. [21, Thm. 1.9.13]). Further, we prove the technical result that addition on a JBW*-predual is jointly sequentially continuous with respect to the abstract measure topology defined in [24]. As a (still technical) consequence this topology is Fréchet-Urysohn on the unit ball.

Some notation. Basic notions and properties not explained here can be found for Banach spaces in [9, 11, 18], and for JBW*-triples in [7, 8], but also, and particularly, in the introductory sections of [22, 23], on which this note relies heavily. We only recall some notation concerning copies of ℓ_1 . Let $r > 0$. We say that elements x_1, \dots, x_n of a Banach space X *span an r -copy of ℓ_1^n* if

$$\sum_{j=1}^n |\alpha_j| \geq \left\| \sum_{j=1}^n \alpha_j x_j \right\| \geq r \sum_{j=1}^n |\alpha_j|,$$

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for all scalars α_j . Let (x_j) be a sequence in X . If (δ_m) is a sequence in $[0, 1[$ tending to zero and $\sum_{j \geq m} |\alpha_j| \geq \left\| \sum_{j \geq m} \alpha_j x_j \right\| \geq (1 - \delta_m) r \sum_{j \geq m} |\alpha_j|$, for all $m \in \mathbb{N}$ and all scalars α_j we say that (x_j) *spans ℓ_1 almost r -isomorphically*. When the latter holds for $r = 1$ we say that (x_j) *spans ℓ_1 almost isometrically*. We say that the sequence (x_j) *spans the spaces ℓ_1^n r -uniformly* (or simply *spans ℓ_1^n 's uniformly*) if for each natural n there are x_{j_1}, \dots, x_{j_n} spanning ℓ_1^n r -isomorphically. If (x_j) spans the spaces ℓ_1^n $(1 - \varepsilon)$ -uniformly for each $\varepsilon > 0$ we say that (x_j) *spans ℓ_1^n 's uniformly almost isometrically*. A block sequence (y_i) of (x_n) is defined by $y_i = \sum_{j \in A_i} \alpha_j x_j$, where the A_i are finite pairwise disjoint subsets of \mathbb{N} and the α_j are scalars.

2. SEQUENCES SPANNING THE SPACES ℓ_1^n IN THE PREDUAL OF A JBW*-TRIPLE

It is known that a Banach space contains the ℓ_1^n 's uniformly almost isometrically if it contains the ℓ_1^n 's uniformly (see e.g. [25, Lemma 1]). We prove the following quantitative version only for lack of an exact reference.

LEMMA 2.1. *If a sequence (x_j) of the closed unit ball of a Banach space spans ℓ_1^n 's r -uniformly (where $r > 0$), then there is a block sequence (y_i) spanning ℓ_1^n 's uniformly almost isometrically; more specifically, the blocks $y_i = \sum_{j \in A_i} \alpha_j x_j$ are such that for each $n \in \mathbb{N}$ the finite sequence $(y_i)_{i \in B_n}$, where $B_n = \left\{ 1 + \frac{(n-1)n}{2}, \dots, \frac{n(n+1)}{2} \right\}$, spans a $(1 - 2^{-n})$ -isomorphic copy of ℓ_1^n , $\|y_i\| = 1$, and $\sum_{j \in A_i} |\alpha_j| < \frac{1 + 2^{-i}}{r}$, for all $i \in \mathbb{N}$.*

Proof. Suppose that (x_j) spans ℓ_1^n 's r -uniformly ($r > 0$). Since the B_n (as in the statement of the lemma) form a partition of $\mathbb{N} = \bigcup_{n \in \mathbb{N}} B_n$ we construct (y_i) by constructing $(y_i)_{i \in B_n}$ inductively over n . Let $n \in \mathbb{N}$, suppose $(y_i)_{i \in B_1}, \dots, (y_i)_{i \in B_{n-1}}$ have been constructed and set $p = \max \bigcup_{q=1}^{n-1} \bigcup_{i \in B_q} A_i$ (with $p = 0$ if $n = 1$). Then the sequence (x_{j+p}) spans ℓ_1^n 's r -uniformly, too.

Set $\rho = r/(1 + 2^{-n(n+1)/2}) < r$. For each natural m we define

$$r_m := \sup_{x_{j_1}, \dots, x_{j_m} \geq p+1} \inf_{\sum_{k=1}^m |\alpha_k| = 1} \left\| \sum_{k=1}^m \alpha_k x_{j_k} \right\|.$$

Then $r_m \geq r$ for all m . The sequence (r_m) is decreasing because for any $\gamma > 0$ and any m there are $x_{j_1}, \dots, x_{j_{m+1}} \geq p+1$ such that $r_{m+1} - \gamma < \inf_{\sum_{k=1}^{m+1} |\alpha_k| = 1} \left\| \sum_{k=1}^{m+1} \alpha_k x_{j_k} \right\| \leq r_m$.

Set $r' = \lim_m r_m$, then we have $r' \geq r$. Choose $\eta > 0$ such that $\frac{r'-\eta}{r'+\eta} > 1 - 2^{-n}$ and $r - \eta > \rho$ and choose m_1 such that $r_{m_1} < r' + \eta$. Set $m_2 = nm_1$. Finally, choose $x_{j_1}, \dots, x_{j_{m_2}}$ such that $r_{m_2} - \eta < \inf_{\sum_{k=1}^{m_2} |\alpha_k|=1} \left\| \sum_{k=1}^{m_2} \alpha_k x_{j_k} \right\|$.

Set $A'_l = \{j_k : k \in \{1 + (l-1)m_1, \dots, lm_1\}\}$, for $l = 1, \dots, n$. The A'_l are pairwise disjoint and are disjoint from the A_i constructed so far (i.e. $i \in \bigcup_{q=1}^{n-1} B_q$ (if $n \geq 2$)). If we set $\tilde{z}_l = \sum_{j \in A'_l} \alpha_j^{(l)} x_j$, for $l \leq n$, where the $\alpha_j^{(l)}$ are such that $\sum_{j \in A'_l} |\alpha_j^{(l)}| = 1$ and

$$\|\tilde{z}_l\| = \inf_{\sum_{j \in A'_l} |\alpha_j|=1} \left\| \sum_{j \in A'_l} \alpha_j x_j \right\|, \text{ then } \|\tilde{z}_l\| \leq r_{m_1} < r' + \eta.$$

Further, we have

$$\inf_{\sum_{l=1}^n |\beta_l|=1} \left\| \sum_{l=1}^n \beta_l \tilde{z}_l \right\| \geq \inf_{\sum_{k=1}^{m_2} |\alpha_k|=1} \left\| \sum_{k=1}^{m_2} \alpha_k x_{j_k} \right\| > r_{m_2} - \eta \geq r' - \eta.$$

In particular, $\|\tilde{z}_l\| > r' - \eta > \rho$. Now we set $z_l = \frac{\tilde{z}_l}{\|\tilde{z}_l\|}$, hence $\|z_l\| = 1$ and $z_l = \sum_{j \in A'_l} \frac{\alpha_j^{(l)}}{\|\tilde{z}_l\|} x_j$

with $\sum_{j \in A'_l} \frac{|\alpha_j^{(l)}|}{\|\tilde{z}_l\|} < 1/\rho = (1 + 2^{-n(n+1)/2})/r \leq (1 + 2^{-i})/r$, for all $i \in B_n$. Moreover, for all scalars β_l ,

$$\sum_{l=1}^n |\beta_l| \geq \left\| \sum_{l=1}^n \beta_l z_l \right\| \geq \frac{r' - \eta}{r' + \eta} \sum_{l=1}^n |\beta_l| \geq (1 - 2^{-n}) \sum_{l=1}^n |\beta_l|,$$

which shows that the z_1, \dots, z_n span a $(1 - 2^{-n})$ -isomorphic copy of ℓ_1^n .

It remains to set $A_i = A'_{i - ((n-1)n/2)}$ and $y_i = z_{i - ((n-1)n/2)}$ for all $i \in B_n$. This ends the induction and the proof. \square

The following proposition is a classical result for L^1 (see e.g. [28]) and follows from [27, Thm. 3.1] for preduals of von Neumann algebras.

PROPOSITION 2.2. *If a sequence in the predual of a JBW*-triple spans ℓ_1^n 's r -uniformly then it admits a subsequence spanning ℓ_1 almost r -isomorphically.*

Proof. Let us first observe a consequence of an extraction lemma of Simons [29]. Given $\varepsilon > 0$, a bounded sequence (x_l) in a Banach space X and a weakly null sequence (x_l^*) in the dual X^* there is a sequence (l_n) in \mathbb{N} such that $\sum_{k \neq n} |x_{l_k}^*(x_{l_n})| < \varepsilon$ for all n . If we apply

this successively for $\varepsilon = 2^{-m}$, $m = 1, 2, \dots$ and pass to the diagonal sequence then we get $\sum_{k \geq m, k \neq n} |x_{l_k}^*(x_{l_n})| < 2^{-m}$ for all m and $n \geq m$.

Suppose (ψ_j) spans ℓ_1^n 's r -uniformly in the predual, W_* , of a JBW*-triple W . We may suppose that $\|\psi_j\| \leq 1$ for all j . Take blocks $\varphi_i = \sum_{j \in A_i} \alpha_j \psi_j$ with

$$(2.1) \quad \sum_{j \in A_i} |\alpha_j| < \frac{1 + 2^{-i}}{r}$$

as given by Lemma 2.1. By [22, Thm. 4.1] there are a subsequence (φ_{i_n}) and a sequence $(\tilde{\varphi}_n)$ of pairwise orthogonal functionals in W_* such that $\|\varphi_{i_n} - \tilde{\varphi}_n\| < 2^{-n}$. Let u_n be the support tripotent of $\tilde{\varphi}_n$ in W . Then $u_n(\tilde{\varphi}_k) = \delta_{k,n} \forall n, k \in \mathbb{N}$. Moreover the u_n are pairwise orthogonal, and thus $\|\sum \theta_n u_n\| \leq 1$ if $|\theta_n| \leq 1$ for all n , and $(u_n) \rightarrow 0$ weakly in W . By (2.1), and since $|u_n(\varphi_{i_n})| > 1 - 2^{-n}$, we obtain $j_n \in A_{i_n}$ such that

$$|u_n(\psi_{j_n})| > \frac{1 - 2^{-n}}{1 + 2^{-n}} r$$

(because otherwise we would have $|u_n(\varphi_{i_n})| \leq \sum_{j \in A_{i_n}} |\alpha_j| |u_n(\psi_j)| \leq \frac{1 + 2^{-i_n}}{r} \frac{1 - 2^{-n}}{1 + 2^{-n}} r \leq 1 - 2^{-n}$). By the observation above we may suppose (by passing to appropriate subsequences) that $\sum_{k \geq m, k \neq n} |u_k(\psi_{j_n})| < 2^{-m}$ for all m and $n \geq m$.

Fix $m \in \mathbb{N}$. Let β_n, θ_n be scalars such that $|\theta_n| = 1$ and $\theta_n u_n(\beta_n \psi_{j_n}) = |\beta_n| |u_n(\psi_{j_n})|$. We have

$$\begin{aligned} \sum_{n \geq m} |\beta_n| &\geq \left\| \sum_{n \geq m} \beta_n \psi_{j_n} \right\| \geq \left| \left(\sum_{k \geq m} \theta_k u_k \right) \left(\sum_{n \geq m} \beta_n \psi_{j_n} \right) \right| \\ &\geq \sum_{n \geq m} |\beta_n| |u_n(\psi_{j_n})| - \sum_{n \geq m} \sum_{k \geq m, k \neq n} |\beta_n| |u_k(\psi_{j_n})| \\ &\geq \frac{1 - 2^{-m}}{1 + 2^{-m}} r \sum_{n \geq m} |\beta_n| - 2^{-m} \sum_{n \geq m} |\beta_n| \geq (1 - \delta_m) r \sum_{n \geq m} |\beta_n| \end{aligned}$$

where $0 < \delta_m = 1 - \left(\frac{1 - 2^{-m}}{1 + 2^{-m}} - \frac{2^{-m}}{r} \right) \rightarrow 0$ as $m \rightarrow \infty$. \square

We briefly recall that a Banach space is said to be *super-reflexive* if all its ultrapowers are reflexive.

COROLLARY 2.3. *Reflexive subspaces of the predual of a JBW*-triple are super-reflexive.*

Proof. Let W_* be the predual of a JBW*-triple W , let U be a closed subspace of W_* . Suppose there is an ultrapower $(U)_\mathcal{U}$ which is not reflexive. This ultrapower is a subspace of the ultrapower $(W_*)_\mathcal{U}$ and the latter is the predual of a JBW*-triple \mathcal{W} ([5, Prop. 5.5]). Hence, by weak sequential completeness of \mathcal{W}_* and Rosenthal's ℓ_1 -theorem, $(U)_\mathcal{U}$ contains an isomorphic copy of ℓ_1 . This copy is finitely representable in U ([16, Thm. 6.3]) hence, by Proposition 2.2, U contains ℓ_1 and is not reflexive. \square

The above result was established by Jarchow [17] in the particular setting of subspaces of the predual of a von Neumann algebra.

DEFINITION 2.4. A Banach space is said to have the *Banach-Saks property* if every bounded sequence (x_n) admits a subsequence (x_{n_k}) such that the Cesàro means $\frac{1}{N} \sum_{k=1}^N x_{n_k}$ (equivalently, the Cesàro means $\frac{1}{N} \sum_{k=1}^N y_k$ of any further subsequence (y_k) of (x_{n_k})) converge in norm.

A Banach space is said to have the *weak Banach-Saks property* (or the Banach-Saks-Rosenthal property) if every weakly null sequence (x_n) admits a subsequence (x_{n_k}) such that the Cesàro means $\frac{1}{N} \sum_{k=1}^N x_{n_k}$ (equivalently, the Cesàro means $\frac{1}{N} \sum_{k=1}^N y_k$ of any further subsequence (y_k) of (x_{n_k})) converge in norm.

The equivalences in the definitions come from a result of Erdős and Magidor [10] (or [4, Prop. II.6.1]), according to which any bounded sequence in a Banach space admits a subsequence such that either all of its subsequences have norm convergent Cesàro means or none of its subsequences has norm convergent Cesàro means. The Banach-Saks property implies reflexivity [4, p. 38] (but not conversely [1], [3, V]) and is implied by superreflexivity [19]. Therefore, by Corollary 2.3, a subspace of a JBW*-predual is reflexive if, and only if, it is super-reflexive if, and only if, it has the (super-)Banach-Saks property. For detailed information on the Banach-Saks property and its variants we refer to [3, 4].

It is due to Szlenk that every weakly convergent sequence in $L_1[0, 1]$ has a subsequence whose arithmetic means are norm convergent (cf. [30], and [9, p. 112]). Szlenk's result on the weak Banach-Saks property has been extended in [6] to duals of C*-algebras, and implicitly to preduals of von Neumann algebras (see the comment in MR0848901). The explicit statement for the predual of a von Neumann algebra N appears in [27, Remark in §5]; cf. also [14, Theorem 5.4] for the case in which N is a finite von Neumann algebra. Next, we establish Szlenk's theorem for JBW*-triple preduals.

COROLLARY 2.5. *The predual of a JBW*-triple has the weak Banach-Saks property.*

Proof. If a Banach space fails the weak Banach-Saks property then, by a result of Rosenthal (cf. [3, Prop. II.1] or [4]) it contains a weakly null sequence which has property (\mathcal{P}_2) in the sense of [3, 4], and therefore spans ℓ_1^n 's uniformly. By Proposition 2.2 this sequence admits a subsequence spanning an isomorphic copy of ℓ_1 which is not possible for a weakly convergent sequence. \square

3. THE ABSTRACT MEASURE TOPOLOGY ON THE PREDUAL OF A JBW*-TRIPLE

A Banach space X is called L-embedded if its bidual can be written $X^{**} = X \oplus_1 X_s$. See the standard reference [15] for details on L-embedded spaces. Preduals of von Neumann algebras and JBW*-triples are L-embedded spaces (cf. [15, Example IV.1.1] and [2, Proposition 3.4]).

In [24, §5] a topology, called the *abstract measure topology*, has been defined on L-embedded Banach spaces X , with the particularity that if X is a commutative or a non-commutative L^1 with finite measure/trace, this topology coincides on (norm) bounded sets with the usual measure topology. The abstract measure topology of an L-embedded space X will be denoted by τ_μ . We recall some properties of τ_μ . It is a sequential topology, which means that sets are closed if they are sequentially closed. Therefore τ_μ is determined by its convergent sequences: a sequence (x_n) converges to x in τ_μ if and only if $(x_n - x)$ is τ_μ -null and a sequence (y_n) is τ_μ -null if and only if each subsequence (y_{n_k}) contains a further subsequence $(y_{n_{k_m}})$ which either is norm null or is semi-normalized (i.e. bounded and $\inf \|y_{n_{k_m}}\| > 0$) and such that $(y_{n_{k_m}}/\|y_{n_{k_m}}\|)$ spans ℓ_1 almost isometrically. In particular, τ_μ -convergent sequences are (norm) bounded. (Note that in [24] the definition of 'to span ℓ_1 almost isometrically' differs slightly from ours but coincides with ours for normalized sequences.) It is not known whether τ_μ is Hausdorff but convergent sequences have unique limits (cf. [24, Proof of Theorem 5.2]). By definition addition is separately continuous with respect to τ_μ , but it is an open problem whether addition is sequentially jointly continuous or even jointly continuous on an L-embedded space X (cf. [24, Question 2]). It can be shown from the results in [27], that addition is τ_μ -sequentially continuous on the unit ball of a von Neumann algebra predual (see [24, Comments in Question 2]). In this section we shall enlarge the examples list by showing that a similar statement remains true for JBW*-triple preduals.

Bounded sets of an L-embedded space satisfy a mild form of sequential compactness with respect to τ_μ : a bounded sequence admits convex blocks that τ_μ -converge [24, Thm. 6.1]. We say that an L-embedded Banach space has the Komlós property if each bounded sequence admits a subsequence such that the Cesàro means of any further subsequence converge with respect to τ_μ (to the same limit). This property is strictly stronger than the just described convex-block compactness because there are L-embedded Banach spaces failing it [24, Example 6.2]. Komlós shows in [20] that $L^1[0, 1]$ has this property and so do preduals of JBW*-triple preduals as we shall show next.

THEOREM 3.1. *The predual of a JBW*-triple has the Komlós property.*

Proof. The theorem is immediate from the splitting property shown in [23, Thm. 6.1], from the weak Banach-Saks property Corollary 2.5 and from [24, §6, Observation] or [26, p. 637]. \square

We can now explore the τ_μ -sequential continuity of the addition on a JBW*-triple predual.

THEOREM 3.2. *Let W be a JBW*-triple with predual W_* . Let W_* be endowed with its abstract measure topology τ_μ . Then addition is sequentially τ_μ -continuous. Consequently, τ_μ is Fréchet-Urysohn on the closed unit ball of W_* .*

Proof. Let $\varphi_n \xrightarrow{\tau_\mu} \varphi$ and $\psi_n \xrightarrow{\tau_\mu} \psi$ in W_* . Since τ_μ is a topology, in order to show that addition is sequentially τ_μ -continuous it is enough to show that $(\varphi_n + \psi_n)$ contains a subsequence τ_μ -converging to $\varphi + \psi$. By translation invariance of τ_μ we may and do suppose that $\varphi = \psi = 0$. By definition of τ_μ -convergent sequences it remains to show that $(\varphi_n + \psi_n)$ contains a subsequence which either is norm null or is semi-normalized and such that $((\varphi_n + \psi_n)/\|\varphi_n + \psi_n\|)$ spans ℓ_1 almost isometrically. If at least one of the two sequences (φ_n) , (ψ_n) contains a norm null subsequence then we are done [24, §5 Remark (b), page 433]. Therefore, the only interesting case we are left with is the following: all $\varphi_n, \psi_n, \varphi_n + \psi_n$ are $\neq 0$, the limits $\lim \|\varphi_n\|$, $\lim \|\psi_n\|$ and $\lim \|\varphi_n + \psi_n\|$ exist and are > 0 and both $(\varphi_n/\|\varphi_n\|)$ and $(\psi_n/\|\psi_n\|)$ span ℓ_1 almost isometrically.

In order to show that there is a subsequence of $((\varphi_n + \psi_n)/\|\varphi_n + \psi_n\|)$ which spans ℓ_1 almost isometrically it is enough, by [22, Thm. 4.1], to show that $((\varphi_n + \psi_n)/\|\varphi_n + \psi_n\|)$ spans ℓ_1^k 's uniformly, that is, it is enough to show for each $k \in \mathbb{N}$ that if for each $\varepsilon' > 0$ some of the $(\varphi_n + \psi_n)/\|\varphi_n + \psi_n\|$ span a $(1 - \varepsilon')$ -copy of ℓ_1^k then for each $\varepsilon > 0$ some finite selection from $((\varphi_n + \psi_n)/\|\varphi_n + \psi_n\|)$ spans a $(1 - \varepsilon)$ -copy of ℓ_1^{k+1} .

Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Choose $\varepsilon' > 0$ such that $(1 - \varepsilon')^2 > 1 - \varepsilon$. Take $\varphi_{n_1} + \psi_{n_1}, \dots, \varphi_{n_k} + \psi_{n_k}$ such that

$$\sum_{l=1}^k |\alpha_l| \|\varphi_{n_l} + \psi_{n_l}\| \geq \left\| \sum_{l=1}^k \alpha_l (\varphi_{n_l} + \psi_{n_l}) \right\| \geq (1 - \varepsilon') \sum_{l=1}^k |\alpha_l| \|\varphi_{n_l} + \psi_{n_l}\|$$

for all scalars α_l .

The following claim is a consequence of Godefroy's well known way to use the w^* -lower semicontinuity of the norm in L-embedded Banach spaces ([13], [15, IV.2.2]) (as it has been used, for example, throughout [24]). The proof is inserted here for the reader's convenience.

Claim. Let $F \subset X$ be a finite dimensional subspace of an L-embedded Banach space X and let (x_n) be a sequence such that $(x_n/\|x_n\|)$ spans ℓ_1 almost isometrically in X and $\lim_{\mathcal{V}} \|x_n\| > 0$ where \mathcal{V} is a non trivial ultrafilter on \mathbb{N} .

Then, for every $\eta > 0$, there is $U \in \mathcal{V}$ such that

$$(3.1) \quad \|\alpha x + \beta x_n\| \geq (1 - \eta)(|\alpha| \|x\| + |\beta| \|x_n\|) \quad \forall \alpha, \beta \in \mathbb{C}, \forall n \in U, \forall x \in F.$$

Proof of the claim. Let x_s be a w^* -accumulation point of (x_n) along \mathcal{V} . By [24, Lemma 3.2], x_s satisfies $x_s \in X_s$ (where $X_s \subset X^{**}$ is as above) and $\|x_s\| = \lim_{\mathcal{V}} \|x_n\|$ hence $\|\alpha x + \beta x_s\| = |\alpha| \|x\| + |\beta| \|x_s\|$ for all $\alpha, \beta \in \mathbb{C}$, $x \in X$.

Given a fixed $x \in F$, $\|x\| = 1$ and finitely many $(\alpha, \beta) \in \ell_1^2$, from $\liminf \left\| \alpha x + \beta \frac{x_n}{\|x_n\|} \right\| \geq \left\| \alpha x + \beta \frac{x_s}{\|x_s\|} \right\|$, we deduce the existence of $U \in \mathcal{V}$ such that

$$(3.2) \quad \left\| \alpha x + \beta \frac{x_n}{\|x_n\|} \right\| \geq (1 - \eta/3)(|\alpha| + |\beta|),$$

for all the (α, β) chosen above and $n \in U$. If these finitely many (α, β) are chosen to form an $\eta/3$ -net of the unit sphere of ℓ_1^2 , then we get (3.2) for all $\alpha, \beta \in \mathbb{C}$ and $n \in U$, but with $2\eta/3$ instead of $\eta/3$. Finally, if one repeats the same reasoning for finitely many x which form an $\eta/3$ -net of the unit sphere of F , then the conclusion of the claim follows.

Let F be the k -dimensional subspace of W_* spanned by the $\varphi_{n_1} + \psi_{n_1}, \dots, \varphi_{n_k} + \psi_{n_k}$. We apply the claim to $X = W_*$ with $\eta = 1/m$, $m \in \mathbb{N}$, which gives a subsequence (φ_{n_m}) such that $\|\alpha\xi + \beta\varphi_{n_m}\| \geq (1 - 1/m)(|\alpha| \|\xi\| + |\beta| \|\varphi_{n_m}\|)$ for all $\alpha, \beta \in \mathbb{C}$, $\xi \in F$. By the same argument applied to the corresponding subsequence of (ψ_n) we may and will assume that also $\|\alpha\xi + \beta\psi_{n_m}\| \geq (1 - 1/m)(|\alpha| \|\xi\| + |\beta| \|\psi_{n_m}\|)$ for all $\alpha, \beta \in \mathbb{C}$, $\xi \in F$.

Now we consider $\widehat{\xi} = [\xi]_{\mathcal{U}}$ for $\xi \in F$, $\widetilde{\varphi} = [\varphi_{n_m}]_{\mathcal{U}}$ and $\widetilde{\psi} = [\psi_{n_m}]_{\mathcal{U}}$ in the ultrapower $(W_*)_{\mathcal{U}}$ where \mathcal{U} is non trivial ultrafilter on \mathbb{N} . It is known that $(W_*)_{\mathcal{U}}$ is the predual of a JBW*-triple \mathcal{W} (cf. [5, Prop. 5.5]). The functionals $\widehat{\xi}$ and $\widetilde{\varphi}$ are L-orthogonal ($\widehat{\xi} \perp_L \widetilde{\varphi}$) in $\mathcal{W}_* = (W_*)_{\mathcal{U}}$ because

$$\|\alpha\widehat{\xi} + \beta\widetilde{\varphi}\| = \lim_{\mathcal{U}} \|\alpha\xi + \beta\varphi_{n_m}\| = |\alpha| \|\widehat{\xi}\| + |\beta| \|\widetilde{\varphi}\|$$

for all $\xi \in F$. Likewise, $\widehat{\xi} \perp_L \widetilde{\psi}$. It is known that $\widehat{\xi} \perp_L \widetilde{\varphi}$ if and only if $\widehat{\xi} \perp \widetilde{\varphi}$, that is, the support tripotents of $\widehat{\xi}$ and $\widetilde{\varphi}$ in \mathcal{W} are orthogonal (i.e. $s(\widehat{\xi}) \perp s(\widetilde{\varphi})$) (cf. [12, Lemma 2.3]). Similarly we get $s(\widehat{\xi}) \perp_L s(\widetilde{\psi})$. Thus $s(\widetilde{\varphi}), s(\widetilde{\psi}) \in \mathcal{W}_0(s(\widehat{\xi}))$, and hence $\widehat{\xi} \perp \widetilde{\varphi} + \widetilde{\psi}$ (because $\widetilde{\varphi}$ and $\widetilde{\psi}$ lie in $(\mathcal{W}_*)_0(s(\widehat{\xi}))$). We deduce that $\widehat{\xi} \perp_L \widetilde{\varphi} + \widetilde{\psi}$ and

$$\lim_{\mathcal{U}} \|\xi + \beta(\varphi_{n_m} + \psi_{n_m})\| = \|\widehat{\xi} + \beta(\widetilde{\varphi} + \widetilde{\psi})\| = \|\xi\| + |\beta| \lim_{\mathcal{U}} \|\varphi_{n_m} + \psi_{n_m}\|.$$

Similarly as above we see that there is $n_{k+1} > n_k$ such that for all $\beta \in \mathbb{C}$, $\xi \in F$ we have

$$\lim_{\mathcal{U}} \|\xi + \beta(\varphi_{n_m} + \psi_{n_m})\| \geq (1 - \varepsilon') (\|\xi\| + |\beta| \|\varphi_{n_{k+1}} + \psi_{n_{k+1}}\|).$$

Let $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{C}$ and consider $\xi = \sum_{l=1}^k \alpha_l(\varphi_{n_l} + \psi_{n_l}) \in F$. Then

$$\begin{aligned} \left\| \sum_{l=1}^{k+1} \alpha_l(\varphi_{n_l} + \psi_{n_l}) \right\| &= \|\xi + \alpha_{k+1}(\varphi_{n_{k+1}} + \psi_{n_{k+1}})\| \\ &\geq (1 - \varepsilon') \left(((1 - \varepsilon') \sum_{l=1}^k |\alpha_l| \|\varphi_{n_l} + \psi_{n_l}\|) + |\alpha_{k+1}| \|\varphi_{n_{k+1}} + \psi_{n_{k+1}}\| \right) \\ &\geq (1 - \varepsilon) \sum_{l=1}^{k+1} |\alpha_l| \|\varphi_{n_m} + \psi_{n_m}\|. \end{aligned}$$

The last assertion follows from [24, Lemma 5.3]. \square

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